9-8

- Final Friday
- Open Evaluations (> 80% response rate, I'll tell you one type of problem that won't be on the final)

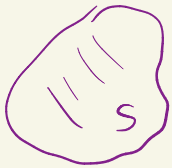
- What we accomplished over the past 6 weeks:

Ch 15: Integration with multiple variables

- Double and Triple Integrals

$\iint dA$ $\iiint dV$

$\text{Area}(S) = \iint_S dA$



- Fubini's theorem:

If $f(x,y)$ continuous, then

$$\iint_R f\,dA = \iint_R f\,dx\,dy = \iint_R f\,dy\,dx$$

- Polar, Cylindrical, and spherical coordinates

captures chain rule $du\,dv = \det J\,dx\,dy$

- Transformations (change of coordinates); Jacobian determinant
- Applications: Center of mass and moment of inertia

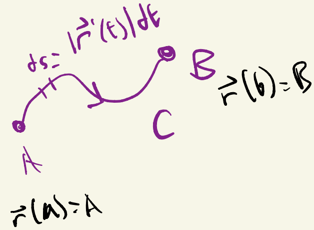
(no parallel axis thm)

appear in physics, probability (expected value + moments) statistics

Ch. 13:

(same methods work in higher dimensions)

- Curves in 2D and 3D space
  - Parameterization $\vec{r}(t)$



should be able to do these!

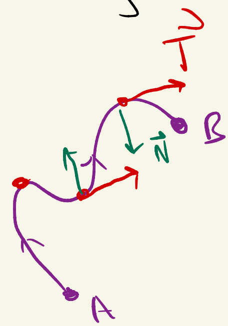
- Lines, circles, and graphs $y=f(x)$

- Arc length of a curve C:

$$\mathrm{Len}(C) = \int ds$$

$$= \int_{t=a}^{t=b} |\vec{r}\,'(t)|\, dt$$

- Orientation of a curve: which direction is increasing, and which direction is decreasing?



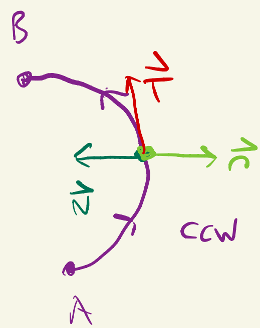
- unit Tangent:

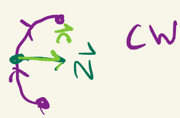
$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

"unit vector in direction of velocity"

- principal unit normal

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$





always have: $\vec{n} = \pm \vec{N}$

warning: unit normal $\vec{n}$ for our line integrals (which comes from orientation in $\mathbb{R}^3$) is convention "CCW rotation = outward normal"

our def: $\vec{n} = \vec{T} \times \hat{k}$

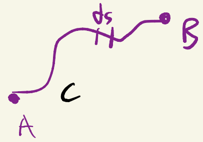
- Curvature

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

Ch 16

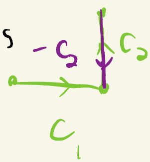
- Line integrals of scalar functions: C an oriented curve

$$\int_C \underbrace{f(x,y,z)}_{\text{scalar functions}}\, ds$$



- Additivity:

$$\int_{C_1} f\,ds = \int_{C_1+C_2-C_2} f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds - \int_{C_2} f\,ds$$



- Calculate line integral: parameterization

$$\int_{t=a}^{t=b} f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\, dt$$

- Vector fields and line integrals

"assign a vector to every point"
vector-valued function

$$\underbrace{\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}}_{\text{vector field}}$$

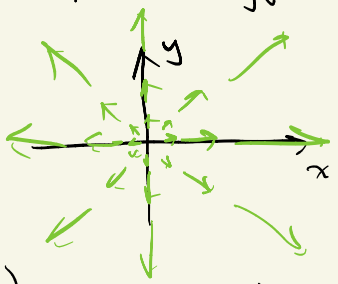
M, N, P are scalar functions

Important vector fields:

- ∇f for function $f(x,y,z)$

- Examples:

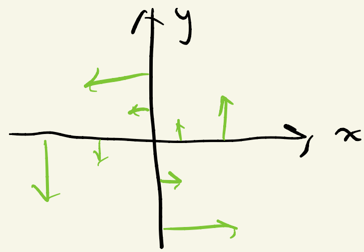
$\vec{F} = x\hat{i} + y\hat{j}$



(source!) - $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} > 0$

- Also happens to be a gradient field

$\vec{F} = -y\hat{i} + x\hat{j}$



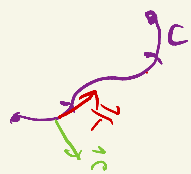
- $\operatorname{curl} \vec{F} = \nabla \times \vec{F} \neq 0$

means by component test that $\vec{F}$ not conservative

- Vector fields give rise to two important line integrals:

$$\text{flow/work} = \int_C \vec{F} \cdot \underbrace{\vec{T}}_{T}\, ds = \int_C \vec{F} \cdot d\vec{r}$$

vector field · vector field = scalar function



special case: $\oint_C \vec{F} \cdot \vec{T}\, ds$ circulation

$$\text{flux} = \int_C \vec{F} \cdot \vec{n}\, ds$$

if C is oriented CCW $\vec{n}$ = unit outward normal

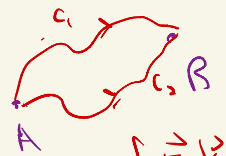
Alt: differential forms

$$\int_C \vec{F} \cdot d\vec{r} = \int M\,dx + N\,dy + P\,dz$$

$$\int_C \vec{F} \cdot \vec{n}\, ds = \int M\,dy - N\,dx$$

If D is simply connected (eg. $\mathbb{R}^2, \mathbb{R}^3, S^2$-sphere)
and $\vec{F}$ is a smooth vector field,

$\vec{F} = \nabla f$ on D $\iff$ $\vec{F}$ conservative on D (path independent) $\iff$ **Loop property:** $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every loop C



$$\int_{c_1} \vec{F} \cdot d\vec{r} = \int_{c_2} \vec{F} \cdot d\vec{r}$$

. Note on simply connected:

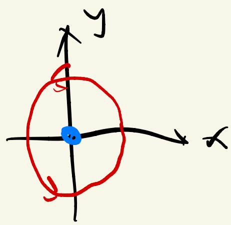
$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{k}$ has singularity at $(0,0)$.

Domain $D = \mathbb{R}^2 - (0,0)$ (punctured plane).

Not simply connected.

$\text{curl}\, \vec{F} = 0$ on D,

but $\oint_{\text{unit circle}} \vec{F} \cdot d\vec{r} = 2\pi$ ← disobeys loop property



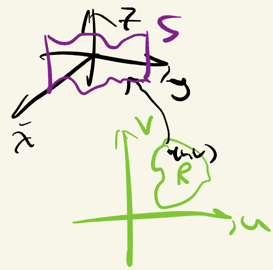
Application: control theory, circuits, mechanical circuits, signal processing, anything with complex numbers (residue theory)

. curl/component test:

$$\nabla \times \vec{F} = 0 \iff \vec{F} \text{ conservative}$$

. If $\vec{F}$ conservative, then we can find a potential f.

Surfaces and areas: $G(x,y,z)$ scalar function



- Parametrization:

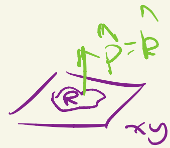
$$\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$$

$$(u,v)\in R$$

$$\iint_S G\,d\sigma = \iint_R G(\vec{r}(u,v))\cdot \underbrace{|\vec{r}_u \times \vec{r}_v|\,du\,dv}_{d\sigma}$$

· Planes, spheres, and graphs $z = f(x,y)$ → same as explicit form

· Implicit



S is level set of $F(x,y,z)$, scalar function

(S defined by $F(x,y,z)=C$)

If $\nabla F\cdot\hat{p}\neq 0$ ($\hat{p}$ is normal to region of integration R)

$$\iint_S G\,d\sigma = \iint_R G(x,y,z)\,\underbrace{\frac{|\nabla F|}{|\nabla F\cdot\hat{p}|}\,dA}_{\text{"}d\sigma}$$

need to use equation to find

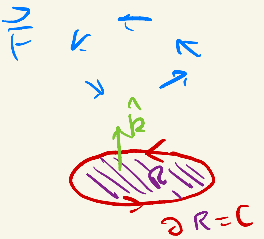
· Explicit:

If $f(x,y)=z$ (graph!)

$$\iint_S G\,d\sigma = \iint_R G(x,y,f(x,y))\,\underbrace{\sqrt{f_x^2+f_y^2+1}\,dx\,dy}_{=d\sigma}$$

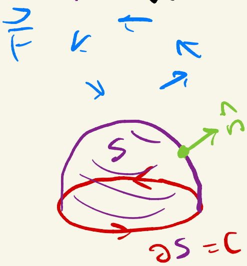
Stokes / Divergence & Green's theorems

- Stoke's theorem extends Green's curl-circulation theorem to <u>any</u> surface R with boundary $\partial R = C$:



Green's (tangential)

$$\iint_R (\nabla \times \vec{F}) \cdot \hat{k} \, dA = \oint_{\partial R} \vec{F} \cdot d\vec{r}$$



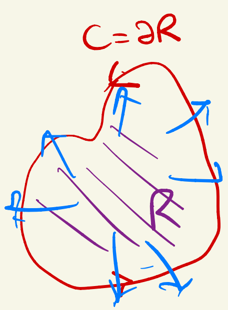
Stokes' Theorem

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, d\sigma = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

Green's Thm (normal)

$$\iint_R (\nabla \cdot \vec{F}) \, dx\, dy = \oint_{\partial R} \vec{F} \cdot \vec{n} \, ds$$

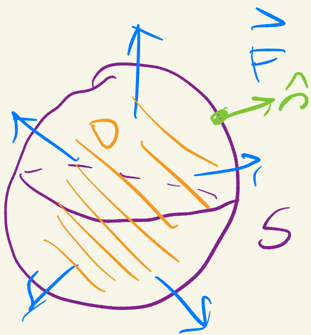
Total divergence inside = outward flux boundary



Divergence Theorem

$$\iiint_D \nabla \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot \hat{n} \, d\sigma$$

Total divergence inside = outward flux boundary



- Should be comfortable with...

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

not really a vector... but practically it is a vector

$(\nabla \cdot \nabla) \times \vec{F}$ is nonsense

"scalar" vector

can't cross

Fact:

$$\operatorname{div}(\operatorname{curl} \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

"divergence of curl is 0"

$$\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) = \vec{0}$$

"curl of gradient is $\vec{0}$"

(component test)

Cool Application: Maxwell's equations (Electromagnetism)

$\vec{E}$ = electric field
$\vec{B}$ = magnetic field

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$ (ρ ← charge density) (Gauss's law)

$$\nabla \cdot \vec{B} = 0$$ (no magnetic monopoles/charges)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ (Faraday's law)

$$\nabla \times \vec{B} = \mu_0\left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$ (Ampere-Maxwell's law)

$\vec{J}$ ↑ current density

Electromagnetic radiation (e.g. light) comes from this interplay of
curling $\vec{E} \Rightarrow$ changing $\vec{B}$
curling $\vec{B} \Rightarrow$ changing $\vec{E}$

. same principle for electric motors

$\Updownarrow$ Stokes/Divergence theorems

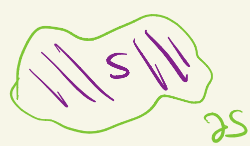
div. Thm:

$$\oint\!\!\oint_{\partial V} \vec{E} \cdot \vec{n}\, dA = \frac{Q}{\varepsilon}$$ (Q ← total charge in V)

$$\oint\!\!\oint_{\partial V} \vec{B} \cdot \vec{n}\, dA = 0$$

stokes:

$$\oint_{\partial S} \vec{E} \cdot d\vec{r} = -\frac{\partial(\text{Flux } \vec{B})}{\partial t}$$



$$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \mu_0\left(I_S + \varepsilon_0 \frac{\partial(\text{Flux}(\vec{E}))}{\partial t}\right)$$

I_S ↑ current